

# Algebraic Number Theory

(PARI-GP version 2.13.4)

## Binary Quadratic Forms

create  $ax^2 + bxy + cy^2$  (distance  $d$ )      **qfb**( $a, b, c, \{d\}$ )  
reduce  $x$  ( $s = \sqrt{D}$ ,  $l = \lfloor s \rfloor$ )      **qfbred**( $x, \{flag\}, \{D\}, \{l\}, \{s\}$ )  
return  $[y, g]$ ,  $g \in \text{SL}_2(\mathbf{Z})$ ,  $y = g \cdot x$  reduced      **qfbreds12**( $x$ )  
composition of forms       $x*y$  or **qfbnucomp**( $x, y, l$ )  
 $n$ -th power of form       $x^n$  or **qfbnupow**( $x, n$ )  
composition without reduction      **qfbcomprow**( $x, y$ )  
 $n$ -th power without reduction      **qfbpowrow**( $x, n$ )  
prime form of disc.  $x$  above prime  $p$       **qfbprimeform**( $x, p$ )  
class number of disc.  $x$       **qfbclassno**( $x$ )  
Hurwitz class number of disc.  $x$       **qfbhclassno**( $x$ )  
solve  $Q(x, y) = n$  in integers      **qfbsolve**( $Q, n$ )

## Quadratic Fields

quadratic number  $\omega = \sqrt{x}$  or  $(1 + \sqrt{x})/2$       **quadgen**( $x$ )  
minimal polynomial of  $\omega$       **quadpoly**( $x$ )  
discriminant of  $\mathbf{Q}(\sqrt{x})$       **quaddisc**( $x$ )  
regulator of real quadratic field      **quadregulator**( $x$ )  
fundamental unit in real  $\mathbf{Q}(\sqrt{D})$       **quadunit**( $D, \{w\}$ )  
class group of  $\mathbf{Q}(\sqrt{D})$       **quadclassunit**( $D, \{flag\}, \{t\}$ )  
Hilbert class field of  $\mathbf{Q}(\sqrt{D})$       **quadhilbert**( $D, \{flag\}$ )  
... using specific class invariant ( $D < 0$ )      **polclass**( $D, \{inv\}$ )  
ray class field modulo  $f$  of  $\mathbf{Q}(\sqrt{D})$       **quadrays**( $D, f, \{flag\}$ )

## General Number Fields: Initializations

The number field  $K = \mathbf{Q}[X]/(f)$  is given by irreducible  $f \in \mathbf{Q}[X]$ . We denote  $\theta = \bar{X}$  the canonical root of  $f$  in  $K$ . A  $nf$  structure contains a maximal order and allows operations on elements and ideals. A  $bnf$  adds class group and units. A  $bnr$  is attached to ray class groups and class field theory. A  $rnf$  is attached to relative extensions  $L/K$ .

init number field structure  $nf$       **nfinit**( $f, \{flag\}$ )  
known integer basis  $B$       **nfinit**( $[f, B]$ )  
order maximal at  $vp = [p_1, \dots, p_k]$       **nfinit**( $[f, vp]$ )  
order maximal at all  $p \leq P$       **nfinit**( $[f, P]$ )  
certify maximal order      **nfcertify**( $nf$ )

### nf members:

a monic  $F \in \mathbf{Z}[X]$  defining  $K$        $nf.pol$   
number of real/complex places       $nf.r1/r2/sign$   
discriminant of  $nf$        $nf.disc$   
primes ramified in  $nf$        $nf.p$   
 $T_2$  matrix       $nf.t2$   
complex roots of  $F$        $nf.roots$   
integral basis of  $\mathbf{Z}_K$  as powers of  $\theta$        $nf.zk$   
different/codifferent       $nf.diff, nf.codiff$   
index  $[\mathbf{Z}_K : \mathbf{Z}[X]/(F)]$        $nf.index$   
recompute  $nf$  using current precision      **nfnewprec**( $nf$ )  
init relative  $rnf$   $L = K[Y]/(g)$       **rnfinit**( $nf, g$ )  
init  $bnf$  structure      **bnfinit**( $f, l$ )

### bnf members:

same as  $nf$ , plus  
underlying  $nf$        $bnf.nf$   
class group, regulator       $bnf.clgp, bnf.reg$   
fundamental/torsion units       $bnf.fu, bnf.tu$   
add  $S$ -class group and units, yield  $bnfS$       **bnfsunit**( $bnf, S$ )

init class field structure  $bnr$       **bnrinit**( $bnf, m, \{flag\}$ )  
**bnr members:** same as  $bnf$ , plus  
underlying  $bnf$        $bnr.bnf$   
big ideal structure       $bnr.bid$   
modulus  $m$        $bnr.mod$   
structure of  $(\mathbf{Z}_K/m)^*$        $bnr.zkst$

## Fields, subfields, embeddings

**Defining polynomials, embeddings**  
smallest poly defining  $f = 0$  (slow)      **polredabs**( $f, \{flag\}$ )  
small poly defining  $f = 0$  (fast)      **polredbest**( $f, \{flag\}$ )  
random Tschirnhausen transform of  $f$       **poltschirnhaus**( $f$ )  
 $\mathbf{Q}[t]/(f) \subset \mathbf{Q}[t]/(g)$  ? Isomorphic?      **nfisincl**( $f, g$ ), **nfisisom**  
reverse polmod  $a = A(t) \bmod T(t)$       **modreverse**( $a$ )  
compositum of  $\mathbf{Q}[t]/(f)$ ,  $\mathbf{Q}[t]/(g)$       **polcompositum**( $f, g, \{flag\}$ )  
compositum of  $K[t]/(f)$ ,  $K[t]/(g)$       **nfcompositum**( $nf, f, g, \{flag\}$ )  
splitting field of  $K$  (degree divides  $d$ )      **nfsplitting**( $nf, \{d\}$ )  
signs of real embeddings of  $x$       **nfeltsign**( $nf, x, \{pl\}$ )  
complex embeddings of  $x$       **nfeltembed**( $nf, x, \{pl\}$ )  
 $T \in K[t]$ , # of real roots of  $\sigma(T) \in R[t]$       **nfpolsturm**( $nf, T, \{pl\}$ )

### Subfields, polynomial factorization

subfields (of degree  $d$ ) of  $nf$       **nfsubfields**( $nf, \{d\}$ )  
maximal subfields of  $nf$       **nfsubfieldsmax**( $nf$ )  
maximal CM subfield of  $nf$       **nfsubfieldscm**( $nf$ )  
 $d$ -th degree subfield of  $\mathbf{Q}(\zeta_n)$       **polsubcyclo**( $n, d, \{v\}$ )  
roots of unity in  $nf$       **nfroots of 1**( $nf$ )  
roots of  $g$  belonging to  $nf$       **nfroots**( $nf, g$ )  
factor  $g$  in  $nf$       **nffactor**( $nf, g$ )

### Linear and algebraic relations

poly of degree  $\leq k$  with root  $x \in \mathbf{C}$       **algdep**( $x, k$ )  
alg. dep. with pol. coeffs for series  $s$       **seralgdep**( $s, x, y$ )  
small linear rel. on coords of vector  $x$       **lindep**( $x$ )

## Basic Number Field Arithmetic (nf)

Number field elements are **t\_INT**, **t\_FRAC**, **t\_POL**, **t\_POLMOD**, or **t\_COL** (on integral basis  $nf.zk$ ).

### Basic operations

$x + y$       **nfeltadd**( $nf, x, y$ )  
 $x \times y$       **nfeltmul**( $nf, x, y$ )  
 $x^n$ ,  $n \in \mathbf{Z}$       **nfeltpow**( $nf, x, n$ )  
 $x/y$       **nfeltdiv**( $nf, x, y$ )  
 $q = x \setminus y := \text{round}(x/y)$       **nfeltdivu**( $nf, x, y$ )  
 $r = x \% y := x - (x \setminus y)y$       **nfeltmod**( $nf, x, y$ )  
...  $[q, r]$  as above      **nfeltdivrem**( $nf, x, y$ )  
reduce  $x$  modulo ideal  $A$       **nfeltreduce**( $nf, x, A$ )  
absolute trace  $\text{Tr}_{K/\mathbf{Q}}(x)$       **nfelttrace**( $nf, x$ )  
absolute norm  $N_{K/\mathbf{Q}}(x)$       **nfeltnorm**( $nf, x$ )

### Multiplicative structure of $K^*$ ; $K^*/(K^*)^n$

valuation  $v_p(x)$       **nfeltval**( $nf, x, p$ )  
... write  $x = \pi^{v_p(x)}y$       **nfeltval**( $nf, x, p, \&y$ )  
quadratic Hilbert symbol (at  $p$ )      **nfhilbert**( $nf, a, b, \{p\}$ )  
 $b$  such that  $xb^n = v$  is small      **idealredmodpower**( $nf, x, n$ )

## Maximal order and discriminant

integral basis of field  $\mathbf{Q}[x]/(f)$       **nfbasis**( $f$ )  
field discriminant of  $\mathbf{Q}[x]/(f)$       **nfdisc**( $f$ )  
... and factorization      **nfdiscfactors**( $f$ )  
express  $x$  on integer basis      **nfalgtobasis**( $nf, x$ )  
express element  $x$  as a polmod      **nfbasistoalg**( $nf, x$ )

## Dedekind Zeta Function $\zeta_K$ , Hecke $L$ series

$R = [c, w, h]$  in initialization means we restrict  $s \in \mathbf{C}$  to domain  $|\Re(s) - c| < w$ ,  $|\Im(s)| < h$ ;  $R = [w, h]$  encodes  $[1/2, w, h]$  and  $[h]$  encodes  $R = [1/2, 0, h]$  (critical line up to height  $h$ ).

$\zeta_K$  as Dirichlet series,  $N(I) < b$       **dirzetak**( $nf, b$ )  
init  $\zeta_K^{(k)}(s)$  for  $k \leq n$       **L = lfunit**( $bnf, R, \{n = 0\}$ )  
compute  $\zeta_K(s)$  ( $n$ -th derivative)      **lfun**( $L, s, \{n = 0\}$ )  
compute  $\Lambda_K(s)$  ( $n$ -th derivative)      **lfunlambda**( $L, s, \{n = 0\}$ )

init  $L_K^{(k)}(s, \chi)$  for  $k \leq n$       **L = lfunit**( $[bnr, chi], R, \{n = 0\}$ )  
compute  $L_K(s, \chi)$  ( $n$ -th derivative)      **lfun**( $L, s, \{n\}$ )  
Artin root number of  $K$       **bnrrootnumber**( $bnr, chi, \{flag\}$ )  
 $L(1, \chi)$ , for all  $\chi$  trivial on  $H$       **bnrL1**( $bnr, \{H\}, \{flag\}$ )

## Class Groups & Units (bnf, bnr)

Class field theory data  $a_1, \{a_2\}$  is usually  $bnr$  (ray class field),  $bnr, H$  (congruence subgroup) or  $bnr, \chi$  (character on **bnr.clgp**). Any of these define a unique abelian extension of  $K$ .

units /  $S$ -units      **bnfunits**( $bnf, \{S\}$ )  
remove GRH assumption from  $bnf$       **bnfcertify**( $bnf$ )  
expo. of ideal  $x$  on class gp      **bnfisprincipal**( $bnf, x, \{flag\}$ )  
expo. of ideal  $x$  on ray class gp      **bnrisprincipal**( $bnr, x, \{flag\}$ )  
expo. of  $x$  on fund. units      **bnfisunit**( $bnf, x$ )  
... on  $S$ -units,  $U$  is **bnfunits**( $bnf, S$ )      **bnfisunit**( $bnfs, U$ )  
signs of real embeddings of  $bnf.fu$       **bnfsignunit**( $bnf$ )  
narrow class group      **bnfnarrow**( $bnf$ )

## Class Field Theory

ray class number for modulus  $m$       **bnrclassno**( $bnf, m$ )  
discriminant of class field      **bnrdisc**( $a_1, \{a_2\}$ )  
ray class numbers,  $l$  list of moduli      **bnrclassnolist**( $bnf, l$ )  
discriminants of class fields      **bnrdiscclst**( $bnf, l, \{arch\}, \{flag\}$ )  
decode output from **bnrdiscclst**      **bnfdecodemodule**( $nf, fa$ )  
is modulus the conductor?      **bnrisconductor**( $a_1, \{a_2\}$ )  
is class field ( $bnr, H$ ) Galois over  $K^G$       **bnrisgalois**( $bnr, G, H$ )  
action of automorphism on **bnr.gen**      **bnrgaloismatrix**( $bnr, aut$ )  
apply **bnrgaloismatrix**  $M$  to  $H$       **bnrgaloisapply**( $bnr, M, H$ )  
characters on **bnr.clgp** s.t.  $\chi(g_i) = e(v_i)$       **bnrchar**( $bnr, g, \{v\}$ )  
conductor of character  $\chi$       **bnrconductor**( $bnr, chi$ )  
conductor of extension      **bnrconductor**( $a_1, \{a_2\}, \{flag\}$ )  
conductor of extension  $K[Y]/(g)$       **rnfconductor**( $bnf, g$ )  
canonical projection  $\text{Cl}_F \rightarrow \text{Cl}_f, f \mid F$       **bnrmmap**  
Artin group of extension  $K[Y]/(g)$       **rnfnormgroup**( $bnr, g$ )  
subgroups of  $bnr$ , index  $\leq b$       **subgrouplist**( $bnr, b, \{flag\}$ )  
class field defined by  $H \subset \text{Cl}_f$       **bnrclassfield**( $bnr, H$ )  
... low level equivalent, prime degree      **rnfkummer**( $bnr, H$ )  
same, using Stark units (real field)      **bnrstark**( $bnr, sub, \{flag\}$ )  
is a an  $n$ -th power in  $K_v$  ?      **nfislocalpower**( $nf, v, a, n$ )  
cyclic  $L/K$  satisf. local conditions      **nfgrunwaldwang**( $nf, P, D, pl$ )

Logarithmic class group

logarithmic $\ell$ -class group	<code>bnflog(<i>bnf</i>, <math>\ell</math>)</code>
$[\tilde{e}(F_v/Q_p), \tilde{f}(F_v/Q_p)]$	<code>bnflogcoef(<i>bnf</i>, <i>pr</i>)</code>
$\exp \deg_F(A)$	<code>bnflogdegree(<i>bnf</i>, <i>A</i>, <math>\ell</math>)</code>
is $\ell$ -extension $L/K$ locally cyclotomic	<code>rnfislocalcyclo(<i>rnf</i>)</code>

**Ideals:** elements, primes, or matrix of generators in HNF

is $id$ an ideal in $nf$ ?	<code>nfisideal(<i>nf</i>, <i>id</i>)</code>
is $x$ principal in $bnf$ ?	<code>bnfisprincipal(<i>bnf</i>, <i>x</i>)</code>
give $[a, b]$ , s.t. $a\mathbf{Z}_K + b\mathbf{Z}_K = x$	<code>idealtwoelt(<i>nf</i>, <i>x</i>, <math>\{a\}</math>)</code>
put ideal $a$ ( $a\mathbf{Z}_K + b\mathbf{Z}_K$ ) in HNF form	<code>idealhnf(<i>nf</i>, <i>a</i>, <math>\{b\}</math>)</code>
norm of ideal $x$	<code>idealnrm(<i>nf</i>, <i>x</i>)</code>
minimum of ideal $x$ (direction $v$ )	<code>idealmin(<i>nf</i>, <i>x</i>, <i>v</i>)</code>
LLL-reduce the ideal $x$ (direction $v$ )	<code>idealred(<i>nf</i>, <i>x</i>, <math>\{v\}</math>)</code>

Ideal Operations

add ideals $x$ and $y$	<code>idealadd(<i>nf</i>, <i>x</i>, <i>y</i>)</code>
multiply ideals $x$ and $y$	<code>idealmul(<i>nf</i>, <i>x</i>, <i>y</i>, <math>\{flag\}</math>)</code>
intersection of ideal $x$ with $Q$	<code>idealdown(<i>nf</i>, <i>x</i>)</code>
intersection of ideals $x$ and $y$	<code>idealintersect(<i>nf</i>, <i>x</i>, <i>y</i>, <math>\{flag\}</math>)</code>
$n$ -th power of ideal $x$	<code>idealpow(<i>nf</i>, <i>x</i>, <i>n</i>, <math>\{flag\}</math>)</code>
inverse of ideal $x$	<code>idealinu(<i>nf</i>, <i>x</i>)</code>
divide ideal $x$ by $y$	<code>idealdiv(<i>nf</i>, <i>x</i>, <i>y</i>, <math>\{flag\}</math>)</code>
Find $(a, b) \in x \times y$ , $a + b = 1$	<code>idealaddtoone(<i>nf</i>, <i>x</i>, <math>\{y\}</math>)</code>
coprime integral $A, B$ such that $x = A/B$	<code>idealnumden(<i>nf</i>, <i>x</i>)</code>

Primes and Multiplicative Structure

check whether $x$ is a maximal ideal	<code>idealismaximal(<i>nf</i>, <i>x</i>)</code>
factor ideal $x$ in $\mathbf{Z}_K$	<code>idealfactor(<i>nf</i>, <i>x</i>)</code>
expand ideal factorization in $K$	<code>idealfactorback(<i>nf</i>, <i>f</i>, <math>\{e\}</math>)</code>
is ideal $A$ an $n$ -th power ?	<code>idealispower(<i>nf</i>, <i>A</i>, <i>n</i>)</code>
expand elt factorization in $K$	<code>nffactorback(<i>nf</i>, <i>f</i>, <math>\{e\}</math>)</code>
decomposition of prime $p$ in $\mathbf{Z}_K$	<code>idealprimedec(<i>nf</i>, <i>p</i>)</code>
valuation of $x$ at prime ideal $pr$	<code>idealval(<i>nf</i>, <i>x</i>, <i>pr</i>)</code>
weak approximation theorem in $nf$	<code>idealchinese(<i>nf</i>, <i>x</i>, <i>y</i>)</code>
$a \in K$ , s.t. $v_{\mathfrak{p}}(a) = v_{\mathfrak{p}}(x)$ if $v_{\mathfrak{p}}(x) \neq 0$	<code>idealappr(<i>nf</i>, <i>x</i>)</code>
$a \in K$ such that $(a \cdot x, y) = 1$	<code>idealcoprime(<i>nf</i>, <i>x</i>, <i>y</i>)</code>
give $bid$ =structure of $(\mathbf{Z}_K/id)^*$	<code>idealstar(<i>nf</i>, <i>id</i>, <math>\{flag\}</math>)</code>
structure of $(1 + \mathfrak{p})/(1 + \mathfrak{p}^k)$	<code>idealprincipalunits(<i>nf</i>, <i>pr</i>, <i>k</i>)</code>
discrete log of $x$ in $(\mathbf{Z}_K/bid)^*$	<code>ideallog(<i>nf</i>, <i>x</i>, <i>bid</i>)</code>
<b>idealstar</b> of all ideals of norm $\leq b$	<code>ideallist(<i>nf</i>, <i>b</i>, <math>\{flag\}</math>)</code>
add Archimedean places	<code>ideallistarch(<i>nf</i>, <i>b</i>, <math>\{ar\}</math>, <math>\{flag\}</math>)</code>
init <b>modpr</b> structure	<code>nfmodprinit(<i>nf</i>, <i>pr</i>, <math>\{v\}</math>)</code>
project $t$ to $\mathbf{Z}_K/pr$	<code>nfmodpr(<i>nf</i>, <i>t</i>, <i>modpr</i>)</code>
lift from $\mathbf{Z}_K/pr$	<code>nfmodprlift(<i>nf</i>, <i>t</i>, <i>modpr</i>)</code>

Galois theory over Q

conjugates of a root $\theta$ of $nf$	<code>nfgaloisconj(<i>nf</i>, <math>\{flag\}</math>)</code>
apply Galois automorphism $s$ to $x$	<code>nfgaloisapply(<i>nf</i>, <i>s</i>, <i>x</i>)</code>
Galois group of field $\mathbf{Q}[x]/(f)$	<code>polgalois(<i>f</i>)</code>
initializes a Galois group structure $G$	<code>galoisinit(<i>pol</i>, <math>\{den\}</math>)</code>
character table of $G$	<code>galoischartable(<i>G</i>)</code>
conjugacy classes of $G$	<code>galoisconjugacyclasses(<i>G</i>)</code>
$\det(1 - \rho(g)T)$ , $\chi$ character of $\rho$	<code>galoischarpoly(<i>G</i>, <math>\chi</math>, <math>\{o\}</math>)</code>
$\det(\rho(g))$ , $\chi$ character of $\rho$	<code>galoischarpoly(<i>G</i>, <math>\chi</math>, <math>\{o\}</math>)</code>
action of $p$ in nfgaloisconj form	<code>galoispermtpol(<i>G</i>, <math>\{p\}</math>)</code>
identify as abstract group	<code>galoisidentify(<i>G</i>)</code>
export a group for GAP/MAGMA	<code>galoisexport(<i>G</i>, <math>\{flag\}</math>)</code>
subgroups of the Galois group $G$	<code>galoissubgroups(<i>G</i>)</code>
is subgroup $H$ normal?	<code>galoisnormal(<i>G</i>, <i>H</i>)</code>

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subfields from subgroups	<code>galoissubfields(<i>G</i>, <math>\{flag\}</math>, <math>\{v\}</math>)</code>
fixed field	<code>galoisfixedfield(<i>G</i>, <i>perm</i>, <math>\{flag\}</math>, <math>\{v\}</math>)</code>
Frobenius at maximal ideal $P$	<code>idealfrobenius(<i>nf</i>, <i>G</i>, <i>P</i>)</code>
ramification groups at $P$	<code>idealramgroups(<i>nf</i>, <i>G</i>, <i>P</i>)</code>
is $G$ abelian?	<code>galoisisabelian(<i>G</i>, <math>\{flag\}</math>)</code>
abelian number fields/ $\mathbf{Q}$	<code>galoissubcyclo(<i>N</i>, <i>H</i>, <math>\{flag\}</math>, <math>\{v\}</math>)</code>

The galpol package

query the package: polynomial	<code>galoisgetpol(<i>a</i>, <i>b</i>, <math>\{s\}</math>)</code>
... : permutation group	<code>galoisgetgroup(<i>a</i>, <i>b</i>)</code>
... : group description	<code>galoisgetname(<i>a</i>, <i>b</i>)</code>

Relative Number Fields (rnf)

Extension  $L/K$  is defined by  $T \in K[x]$ .

absolute equation of $L$	<code>rnfequation(<i>nf</i>, <i>T</i>, <math>\{flag\}</math>)</code>
is $L/K$ abelian?	<code>rnfisabelian(<i>nf</i>, <i>T</i>)</code>
relative nfaltobasis	<code>rnfaltobasis(<i>rnf</i>, <i>x</i>)</code>
relative nfbasistoalg	<code>rnfbasistoalg(<i>rnf</i>, <i>x</i>)</code>
relative idealhnf	<code>rnfidealhnf(<i>rnf</i>, <i>x</i>)</code>
relative idealmul	<code>rnfidealmul(<i>rnf</i>, <i>x</i>, <i>y</i>)</code>
relative idealtwoelt	<code>rnfidealtwoelt(<i>rnf</i>, <i>x</i>)</code>

Lifts and Push-downs

absolute $\rightarrow$ relative representation for $x$	<code>rnfeltabstorel(<i>rnf</i>, <i>x</i>)</code>
relative $\rightarrow$ absolute representation for $x$	<code>rnfeltreltoabs(<i>rnf</i>, <i>x</i>)</code>
lift $x$ to the relative field	<code>rnfeltup(<i>rnf</i>, <i>x</i>)</code>
push $x$ down to the base field	<code>rnfeltdown(<i>rnf</i>, <i>x</i>)</code>
idem for $x$ ideal: (rnfideal)reltoabs, abstorel, up, down	

Norms and Trace

relative norm of element $x \in L$	<code>rnfeltnorm(<i>rnf</i>, <i>x</i>)</code>
relative trace of element $x \in L$	<code>rnfelttrace(<i>rnf</i>, <i>x</i>)</code>
absolute norm of ideal $x$	<code>rnfidealnrmabs(<i>rnf</i>, <i>x</i>)</code>
relative norm of ideal $x$	<code>rnfidealnrmrel(<i>rnf</i>, <i>x</i>)</code>
solutions of $N_{K/\mathbf{Q}}(y) = x \in \mathbf{Z}$	<code>bnfisintnorm(<i>bnf</i>, <i>x</i>)</code>
is $x \in \mathbf{Q}$ a norm from $K$ ?	<code>bnfisnorm(<i>bnf</i>, <i>x</i>, <math>\{flag\}</math>)</code>
initialize $T$ for norm eq. solver	<code>rnfnorminit(<i>K</i>, <i>pol</i>, <math>\{flag\}</math>)</code>
is $a \in K$ a norm from $L$ ?	<code>rnfnisnorm(<i>T</i>, <i>a</i>, <math>\{flag\}</math>)</code>
initialize $t$ for Thue equation solver	<code>thueinit(<i>f</i>)</code>
solve Thue equation $f(x, y) = a$	<code>thue(<i>t</i>, <i>a</i>, <math>\{sol\}</math>)</code>
characteristic poly. of $a$ mod $T$	<code>rnfcharpoly(<i>nf</i>, <i>T</i>, <i>a</i>, <math>\{v\}</math>)</code>

Factorization

factor ideal $x$ in $L$	<code>rnfidealfactor(<i>rnf</i>, <i>x</i>)</code>
$[S, T]: T_{i,j} \mid S_i$ ; $S$ primes of $K$ above $p$	<code>rnfidealprimedec(<i>rnf</i>, <i>p</i>)</code>

Maximal order  $\mathbf{Z}_L$  as a  $\mathbf{Z}_K$ -module

relative polredbest	<code>rnfpolredbest(<i>nf</i>, <i>T</i>)</code>
relative polredabs	<code>rnfpolredabs(<i>nf</i>, <i>T</i>)</code>
relative Dedekind criterion, prime $pr$	<code>rnfdedekind(<i>nf</i>, <i>T</i>, <i>pr</i>)</code>
discriminant of relative extension	<code>rnfdisc(<i>nf</i>, <i>T</i>)</code>
pseudo-basis of $\mathbf{Z}_L$	<code>rnfpseudobasis(<i>nf</i>, <i>T</i>)</code>

**General  $\mathbf{Z}_K$ -modules:**  $M = [\text{matrix, vec. of ideals}] \subset L$

relative HNF / SNF	<code>nfhnf(<i>nf</i>, <i>M</i>), nfsnf</code>
multiple of det $M$	<code>nfdetint(<i>nf</i>, <i>M</i>)</code>
HNF of $M$ where $d = nfdetint(M)$	<code>nfhnfmod(<i>x</i>, <i>d</i>)</code>
reduced basis for $M$	<code>rnfilllgram(<i>nf</i>, <i>T</i>, <i>M</i>)</code>
determinant of pseudo-matrix $M$	<code>rnfdet(<i>nf</i>, <i>M</i>)</code>
Steinitz class of $M$	<code>rnfstinitz(<i>nf</i>, <i>M</i>)</code>

$\mathbf{Z}_K$ -basis of  $M$  if  $\mathbf{Z}_K$ -free, or 0

$n$ -basis of  $M$ , or  $(n + 1)$ -generating set

is  $M$  a free  $\mathbf{Z}_K$ -module?

Associative Algebras

$A$  is a general associative algebra given by a multiplication table  $mt$  (over  $\mathbf{Q}$  or  $\mathbf{F}_p$ ); represented by  $al$  from algtableinit.

create $al$ from $mt$ (over $\mathbf{F}_p$ )	<code>algtableinit(<i>mt</i>, <math>\{p = 0\}</math>)</code>
group algebra $\mathbf{Q}[G]$ (or $\mathbf{F}_p[G]$ )	<code>algroup(<i>G</i>, <math>\{p = 0\}</math>)</code>
center of group algebra	<code>algroupcenter(<i>G</i>, <math>\{p = 0\}</math>)</code>

Properties

is $(mt, p)$ OK for algtableinit?	<code>algisassociative(<i>mt</i>, <math>\{p = 0\}</math>)</code>
multiplication table $mt$	<code>algmtable(<i>al</i>)</code>
dimension of $A$ over prime subfield	<code>algdim(<i>al</i>)</code>
characteristic of $A$	<code>algchar(<i>al</i>)</code>
is $A$ commutative?	<code>algiscommutative(<i>al</i>)</code>
is $A$ simple?	<code>algisissimple(<i>al</i>)</code>
is $A$ semi-simple?	<code>algissemisimple(<i>al</i>)</code>
center of $A$	<code>algcenter(<i>al</i>)</code>
Jacobson radical of $A$	<code>algradical(<i>al</i>)</code>
radical $J$ and simple factors of $A/J$	<code>algsimpledec(<i>al</i>)</code>

Operations on algebras

create $A/I$ , $I$ two-sided ideal	<code>algquotient(<i>al</i>, <i>I</i>)</code>
create $A_1 \otimes A_2$	<code>algtensor(<i>al1</i>, <i>al2</i>)</code>
create subalgebra from basis $B$	<code>algsubalg(<i>al</i>, <i>B</i>)</code>
quotients by ortho. central idempotents $e$	<code>algcentralproj(<i>al</i>, <i>e</i>)</code>
isomorphic alg. with integral mult. table	<code>algmakeintegral(<i>mt</i>)</code>
prime subalgebra of semi-simple $A$ over $\mathbf{F}_p$	<code>algprimesubalg(<i>al</i>)</code>
find isomorphism $A \cong M_d(\mathbf{F}_q)$	<code>algsplit(<i>al</i>)</code>

Operations on lattices in algebras

lattice generated by cols. of $M$	<code>alglathnf(<i>al</i>, <i>M</i>)</code>
... by the products $xy$ , $x \in lat1$ , $y \in lat2$	<code>alglatmul(<i>al</i>, <i>lat1</i>, <i>lat2</i>)</code>
sum $lat1 + lat2$ of the lattices	<code>alglatadd(<i>al</i>, <i>lat1</i>, <i>lat2</i>)</code>
intersection $lat1 \cap lat2$	<code>alglatinter(<i>al</i>, <i>lat1</i>, <i>lat2</i>)</code>
test $lat1 \subset lat2$	<code>alglatsubset(<i>al</i>, <i>lat1</i>, <i>lat2</i>)</code>
generalized index $(lat2 : lat1)$	<code>alglatindex(<i>al</i>, <i>lat1</i>, <i>lat2</i>)</code>
$\{x \in al \mid x \cdot lat1 \subset lat2\}$	<code>alglatlefttransporter(<i>al</i>, <i>lat1</i>, <i>lat2</i>)</code>
$\{x \in al \mid lat1 \cdot x \subset lat2\}$	<code>alglatrighttransporter(<i>al</i>, <i>lat1</i>, <i>lat2</i>)</code>
test $x \in lat$ (set $c$ = coord. of $x$ )	<code>alglatcontains(<i>al</i>, <i>lat</i>, <i>x</i>, <math>\{&amp;c\}</math>)</code>
element of $lat$ with coordinates $c$	<code>alglatelement(<i>al</i>, <i>lat</i>, <i>c</i>)</code>

Operations on elements

$a + b$ , $a - b$ , $-a$	<code>algadd(<i>al</i>, <i>a</i>, <i>b</i>), algsub, algneg</code>
$a \times b$ , $a^2$	<code>algmul(<i>al</i>, <i>a</i>, <i>b</i>), algsqr</code>
$a^n$ , $a^{-1}$	<code>algpow(<i>al</i>, <i>a</i>, <i>n</i>), alginv</code>
is $x$ invertible ? (then set $z = x^{-1}$ )	<code>alginv(<i>al</i>, <i>x</i>, <math>\{&amp;z\}</math>)</code>
find $z$ such that $x \times z = y$	<code>algdivl(<i>al</i>, <i>x</i>, <i>y</i>)</code>
find $z$ such that $z \times x = y$	<code>algdivr(<i>al</i>, <i>x</i>, <i>y</i>)</code>
does $s$ s.t. $x \times z = y$ exist? (set it)	<code>algisdivl(<i>al</i>, <i>x</i>, <i>y</i>, <math>\{&amp;z\}</math>)</code>
matrix of $v \mapsto x \cdot v$	<code>algtomatrix(<i>al</i>, <i>x</i>)</code>
absolute norm	<code>algnorm(<i>al</i>, <i>x</i>)</code>
absolute trace	<code>algtrace(<i>al</i>, <i>x</i>)</code>
absolute char. polynomial	<code>algcharpoly(<i>al</i>, <i>x</i>)</code>
given $a \in A$ and polynomial $T$ , return $T(a)$	<code>algpoleval(<i>al</i>, <i>T</i>, <i>a</i>)</code>
random element in a box	<code>algrandom(<i>al</i>, <i>b</i>)</code>

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Central Simple Algebras

$A$  is a central simple algebra over a number field  $K$ ; represented by  $al$  from **alginit**;  $K$  is given by a  $nf$  structure.  
create CSA from data           **alginit**( $B, C, \{v\}, \{maxord = 1\}$ )  
multiplication table over  $K$             $B = K, C = mt$   
cyclic algebra ( $L/K, \sigma, b$ )            $B = rnf, C = [sigma, b]$   
quaternion algebra  $(a, b)_K$             $B = K, C = [a, b]$   
matrix algebra  $M_d(K)$             $B = K, C = d$   
local Hasse invariants over  $K$     $B = K, C = [d, [PR, HF], HI]$

Properties

type of  $al$  ( $mt, CSA$ )           **algtype**( $al$ )  
dimension of  $A$  over  $\mathbf{Q}$            **algdim**( $al, 1$ )  
dimension of  $al$  over its center  $K$    **algdim**( $al$ )  
degree of  $A$  ( $= \sqrt{\dim_K A}$ )       **algdegree**( $al$ )  
 $al$  a cyclic algebra ( $L/K, \sigma, b$ ); return  $\sigma$    **algaut**( $al$ )  
...return  $b$                    **algb**( $al$ )  
...return  $L/K$ , as an  $rnf$        **algsplittingfield**( $al$ )  
split  $A$  over an extension of  $K$        **algsplittingdata**( $al$ )  
splitting field of  $A$  as an  $rnf$  over center   **algsplittingfield**( $al$ )  
multiplication table over center       **algrelmultable**( $al$ )  
places of  $K$  at which  $A$  ramifies       **algramifiedplaces**( $al$ )  
Hasse invariants at finite places of  $K$    **alghassef**( $al$ )  
Hasse invariants at infinite places of  $K$    **alghassei**( $al$ )  
Hasse invariant at place  $v$            **alghasse**( $al, v$ )  
index of  $A$  over  $K$  (at place  $v$ )       **algindex**( $al, \{v\}$ )  
is  $al$  a division algebra? (at place  $v$ )   **algisdivision**( $al, \{v\}$ )  
is  $A$  ramified? (at place  $v$ )       **algisramified**( $al, \{v\}$ )  
is  $A$  split? (at place  $v$ )           **algissplit**( $al, \{v\}$ )

Operations on elements

reduced norm                   **algnorm**( $al, x$ )  
reduced trace                  **algtrace**( $al, x$ )  
reduced char. polynomial       **algcharpoly**( $al, x$ )  
express  $x$  on integral basis    **algalgtobasis**( $al, x$ )  
convert  $x$  to algebraic form    **algbasistoalg**( $al, x$ )  
map  $x \in A$  to  $M_d(L)$ ,  $L$  split. field   **algtomatrix**( $al, x$ )

Orders

**Z**-basis of order  $\mathcal{O}_0$            **algbasis**( $al$ )  
discriminant of order  $\mathcal{O}_0$        **algdisc**( $al$ )  
**Z**-basis of natural order in terms  $\mathcal{O}_0$ 's basis   **alginvbasis**( $al$ )