

# Modular forms, modular symbols

(PARI-GP version 2.15.0)

## Modular Forms

### Dirichlet characters

Characters are encoded in three different ways:

- a `t_INT`  $D \equiv 0, 1 \bmod 4$ : the quadratic character  $(D/\cdot)$ ;
- a `t_INTMOD`  $\text{Mod}(m, q)$ ,  $m \in (\mathbf{Z}/q)^*$  using a canonical bijection with the dual group (the Conrey character  $\chi_q(m, \cdot)$ );
- a pair  $[G, \text{chi}]$ , where  $G = \text{znstar}(q, 1)$  encodes  $(\mathbf{Z}/q\mathbf{Z})^* = \sum_{j \leq k} (\mathbf{Z}/d_j\mathbf{Z}) \cdot g_j$  and the vector  $\text{chi} = [c_1, \dots, c_k]$  encodes the character such that  $\chi(g_j) = e(c_j/d_j)$ .

initialize  $G = (\mathbf{Z}/q\mathbf{Z})^*$  `G = znstar(q, 1)`  
convert datum  $D$  to  $[G, \chi]$  `znchar(D)`  
Galois orbits of Dirichlet characters `chargalois(G)`

### Spaces of modular forms

Arguments of the form  $[N, k, \chi]$  give the level weight and nebentypus  $\chi$ ;  $\chi$  can be omitted:  $[N, k]$  means trivial  $\chi$ .

initialize  $S_k^{\text{new}}(\Gamma_0(N), \chi)$  `mfinit([N, k, \chi], 0)`  
initialize  $S_k(\Gamma_0(N), \chi)$  `mfinit([N, k, \chi], 1)`  
initialize  $S_k^{\text{old}}(\Gamma_0(N), \chi)$  `mfinit([N, k, \chi], 2)`  
initialize  $E_k(\Gamma_0(N), \chi)$  `mfinit([N, k, \chi], 3)`  
initialize  $M_k(\Gamma_0(N), \chi)$  `mfinit([N, k, \chi])`  
find eigenforms `mfsplit(M)`  
statistics on self-growing caches `getcache()`

We let  $M = \text{mfinit}(\dots)$  denote a modular space.  
describe the space  $M$  `mfdescribe(M)`  
recover  $(N, k, \chi)$  `mfparams(M)`  
... the space identifier (0 to 4) `mfspace(M)`  
... the dimension of  $M$  over  $\mathbf{C}$  `mfdim(M)`  
... a  $\mathbf{C}$ -basis  $(f_i)$  of  $M$  `mfbasis(M)`  
... a basis  $(F_j)$  of eigenforms `mfeigenbasis(M)`  
... polynomials defining  $\mathbf{Q}(\chi)/(F_j)/\mathbf{Q}(\chi)$  `mffields(M)`

matrix of Hecke operator  $T_n$  on  $(f_i)$  `mfheckemat(M, n)`  
eigenvalues of  $w_Q$  `mfatkineigenvalues(M, Q)`  
basis of period polynomials for weight  $k$  `mferiodpolbasis(k)`  
basis of the Kohnen  $+$ -space `mfkohnenbasis(M)`  
... new space and eigenforms `mfkohneneigenbasis(M, b)`  
isomorphism  $S_k^+(4N, \chi) \rightarrow S_{2k-1}(N, \chi^2)$  `mfkohnenbijection(M)`

Useful data can also be obtained a priori, without computing a complete modular space:

dimension of  $S_k^{\text{new}}(\Gamma_0(N), \chi)$  `mfdim([N, k, \chi])`  
dimension of  $S_k(\Gamma_0(N), \chi)$  `mfdim([N, k, \chi], 1)`  
dimension of  $S_k^{\text{old}}(\Gamma_0(N), \chi)$  `mfdim([N, k, \chi], 2)`  
dimension of  $M_k(\Gamma_0(N), \chi)$  `mfdim([N, k, \chi], 3)`  
dimension of  $E_k(\Gamma_0(N), \chi)$  `mfdim([N, k, \chi], 4)`  
Sturm's bound for  $M_k(\Gamma_0(N), \chi)$  `mfsturm(N, k)`  
 $\Gamma_0(N)$  cosets `mfcosets(N)`  
list of right  $\Gamma_0(N)$  cosets `mfcosets(N)`  
identify coset a matrix belongs to `mftocoset`

### Cusps

a cusp is given by a rational number or  $\infty$ .  
lists of cusps of  $\Gamma_0(N)$  `mfcusps(N)`  
number of cusps of  $\Gamma_0(N)$  `mfnumcusps(N)`  
width of cusp  $c$  of  $\Gamma_0(N)$  `mfcuspswidth(N, c)`  
is cusp  $c$  regular for  $M_k(\Gamma_0(N), \chi)$ ? `mfcuspisregular([N, k, \chi], c)`

### Create an individual modular form

Besides `mfbasis` and `mfeigenbasis`, an individual modular form can be identified by a few coefficients.

modular form from coefficients `mftobasis(mf, vec)`  
There are also many predefined ones:  
Eisenstein series  $E_k$  on  $Sl_2(\mathbf{Z})$  `mfEk(k)`  
Eisenstein-Hurwitz series on  $\Gamma_0(4)$  `mfEH(k)`  
unary  $\theta$  function (for character  $\psi$ ) `mfTheta({\psi})`  
Ramanujan's  $\Delta$  `mfDelta()`  
 $E_k(\chi)$  `mfeisenstein(k, \chi)`  
 $E_k(\chi_1, \chi_2)$  `mfeisenstein(k, \chi_1, \chi_2)`  
eta quotient  $\prod_i \eta(a_{i,1} \cdot z)^{a_{i,2}}$  `mffrometaquo(a)`  
newform attached to ell. curve  $E/\mathbf{Q}$  `mffromell(E)`  
identify an  $L$ -function as a eigenform `mffromlfun(L)`  
 $\theta$  function attached to  $Q > 0$  `mffromqt(Q)`  
trace form in  $S_k^{\text{new}}(\Gamma_0(N), \chi)$  `mftraceform([N, k, \chi])`  
trace form in  $S_k(\Gamma_0(N), \chi)$  `mfttraceform([N, k, \chi], 1)`

### Operations on modular forms

In this section,  $f, g$  and the  $F[i]$  are modular forms  
 $f \times g$  `mfmul(f, g)`  
 $f/g$  `mfddiv(f, g)`  
 $f^n$  `mfpow(f, n)`  
 $f(q)/q^v$  `mfshift(f, v)`  
 $\sum_{i \leq k} \lambda_i F[i]$ ,  $L = [\lambda_1, \dots, \lambda_k]$  `mflinear(F, L)`  
 $f = g?$  `mfisequal(f, g)`  
expanding operator  $B_d(f)$  `mfbd(f, d)`  
Hecke operator  $T_n f$  `mfhecke(mf, f, n)`  
initialize Atkin-Lehner operator  $w_Q$  `mfatkininit(mf, f, Q)`  
... apply  $w_Q$  to  $f$  `mfatkin(w_Q, f)`  
twist by the quadratic char  $(D/\cdot)$  `mftwist(f, D)`  
derivative wrt.  $q \cdot d/dq$  `mfderiv(f)`  
see  $f$  over an absolute field `mfreltoabs(f)`  
Serre derivative  $\left(q \cdot \frac{d}{dq} - \frac{k}{12} E_2\right) f$  `mfderivE2(f)`  
Rankin-Cohen bracket  $[f, g]_n$  `mfbracket(f, g, n)`  
Shimura lift of  $f$  for discriminant  $D$  `mfshimura(mf, f, D)`

### Properties of modular forms

In this section,  $f = \sum_n f_n q^n$  is a modular form in some space  $M$  with parameters  $N, k, \chi$ .  
describe the form  $f$  `mfdescribe(f)`  
 $(N, k, \chi)$  for form  $f$  `mfparams(f)`  
the space identifier (0 to 4) for  $f$  `mfspace(mf, f)`  
 $[f_0, \dots, f_n]$  `mfcoefs(f, n)`  
 $f_n$  `mfcoef(f, n)`  
is  $f$  a CM form? `mfisCM(f)`  
is  $f$  an eta quotient? `mfisetaquo(f)`

Galois rep. attached to all  $(1, \chi)$  eigenforms `mfgaloisotype(M)`  
... single eigenform `mfgaloisotype(M, F)`  
... as a polynomial fixed by  $\text{Ker } \rho_F$  `mfgaloisprojrep(M, F)`  
decompose  $f$  on `mfbasis(M)` `mftobasis(M, f)`  
smallest level on which  $f$  is defined `mfconductor(M, f)`  
decompose  $f$  on  $\oplus S_k^{\text{new}}(\Gamma_0(d))$ ,  $d \mid N$  `mftonew(M, f)`  
valuation of  $f$  at cusp  $c$  `mfcuspsval(M, f, c)`  
expansion at  $\infty$  of  $f|_k \gamma$  `mfslashexpansion(M, f, \gamma, n)`  
 $n$ -Taylor expansion of  $f$  at  $i$  `mftaylor(f, n)`  
all rational eigenforms matching criteria `mfeigensearch`  
... forms matching criteria `mfsearch`

### Forms embedded into $\mathbf{C}$

Given a modular form  $f$  in  $M_k(\Gamma_0(N), \chi)$  its field of definition  $Q(f)$  has  $n = [Q(f) : Q(\chi)]$  embeddings into the complex numbers. If  $n = 1$ , the following functions return a single answer, attached to the canonical embedding of  $f$  in  $\mathbf{C}[[q]]$ ; else a vector of  $n$  results, corresponding to the  $n$  conjugates of  $f$ .

complex embeddings of  $Q(f)$  `mfembed(f)`  
... embed coefs of  $f$  `mfembed(f, v)`  
evaluate  $f$  at  $\tau \in \mathcal{H}$  `mfeval(f, \tau)`  
 $L$ -function attached to  $f$  `lfunmf(mf, f)`  
... eigenforms of new space  $M$  `lfunmf(M)`

### Periods and symbols

The functions in this section depend on  $[Q(f) : Q(\chi)]$  as above.  
initialize symbol  $fs$  attached to  $f$  `mfsymbol(M, f)`  
evaluate symbol  $fs$  on path  $p$  `mfssymboleval(fs, p)`  
Petersson product of  $f$  and  $g$  `mfpetersson(fs, gs)`  
period polynomial of form  $f$  `mpperiodpol(M, fs)`  
period polynomials for eigensymbol  $FS$  `mfmanin(FS)`

## Modular Symbols

Let  $G = \Gamma_0(N)$ ,  $V_k = \mathbf{Q}[X, Y]_{k-2}$ ,  $L_k = \mathbf{Z}[X, Y]_{k-2}$  and  $\Delta = \text{Div}^0(\mathbf{P}^1(\mathbf{Q}))$ . An element of  $\Delta$  is a *path* between cusps of  $X_0(N)$  via the identification  $[b] - [a] \rightarrow$  path from  $a$  to  $b$ , coded by the pair  $[a, b]$  where  $a, b$  are rationals or  $\infty = (1 : 0)$ .

Let  $\mathbf{M}_k(G) = \text{Hom}_G(\Delta, V_k) \simeq H_c^1(X_0(N), V_k)$ ; an element of  $\mathbf{M}_k(G)$  is a  $V_k$ -valued *modular symbol*. There is a natural decomposition  $\mathbf{M}_k(G) = \mathbf{M}_k(G)^+ \oplus \mathbf{M}_k(G)^-$  under the action of the  $*$  involution, induced by complex conjugation. The `msinit` function computes either  $\mathbf{M}_k$  ( $\varepsilon = 0$ ) or its  $\pm$ -parts ( $\varepsilon = \pm 1$ ) and fixes a minimal set of  $\mathbf{Z}[G]$ -generators  $(g_i)$  of  $\Delta$ .

initialize  $M = \mathbf{M}_k(\Gamma_0(N))^\varepsilon$  `msinit(N, k, {\varepsilon = 0})`  
the level  $M$  `msgetlevel(M)`  
the weight  $k$  `msgetweight(M)`  
the sign  $\varepsilon$  `msgetsign(M)`  
Farey symbol attached to  $G$  `mspolygon(M)`  
... attached to  $H < G$  `msfarey(F, inH)`  
 $H \backslash G$  and right  $G$ -action `mscosets(genG, inH)`

$\mathbf{Z}[G]$ -generators  $(g_i)$  and relations for  $\Delta$  `mspathgens(M)`  
decompose  $p = [a, b]$  on the  $(g_i)$  `mspathlog(M, p)`

### Create a symbol

Eisenstein symbol attached to cusp  $c$  `msfromcusp(M, c)`  
cuspidal symbol attached to  $E/\mathbf{Q}$  `msfromell(E)`  
symbol having given Hecke eigenvalues `msfromhecke(M, v, {H})`  
is  $s$  a symbol? `msissymbol(M, s)`

### Operations on symbols

the list of all  $s(g_i)$  `mseval(M, s)`  
evaluate symbol  $s$  on path  $p = [a, b]$  `mseval(M, s, p)`  
Petersson product of  $s$  and  $t$  `mspetersson(M, s, t)`

### Operators on subspaces

An operator is given by a matrix of a fixed  $\mathbf{Q}$ -basis.  $H$ , if given, is a stable  $\mathbf{Q}$ -subspace of  $\mathbf{M}_k(G)$ : operator is restricted to  $H$ .  
matrix of Hecke operator  $T_p$  or  $U_p$  `mshecke(M, p, {H})`  
matrix of Atkin-Lehner  $w_Q$  `msatkinlehner(M, {Q\{H\})`  
matrix of the  $*$  involution `msstar(M, {H})`

Subspaces

A subspace is given by a structure allowing quick projection and restriction of linear operators. Its fist component is a matrix with integer coefficients whose columns for a  $\mathbf{Q}$ -basis. If  $H$  is a Hecke-stable subspace of  $M_k(G)^+$  or  $M_k(G)^-$ , it can be split into a direct sum of Hecke-simple subspaces. To a simple subspace corresponds a single normalized newform  $\sum_n a_n q^n$ .

|  |                                      |
|--|--------------------------------------|
| cuspidal subspace $S_k(G)^\varepsilon$                     | <code>mscuspidal(M)</code>           |
| Eisenstein subspace $E_k(G)^\varepsilon$                   | <code>mseisenstein(M)</code>         |
| new part of $S_k(G)^\varepsilon$                           | <code>msnew(M)</code>                |
| split $H$ into simple subspaces (of $\dim \leq d$ )        | <code>mssplit(M, H, {d})</code>      |
| dimension of a subspace                                    | <code>msdim(M)</code>                |
| $(a_1, \dots, a_B)$ for attached newform                   | <code>msqexpansion(M, H, {B})</code> |
| $\mathbf{Z}$ -structure from $H^1(G, L_k)$ on subspace $A$ | <code>mslattice(M, A)</code>         |

Overconvergent symbols and  $p$ -adic  $L$  functions

Let  $M$  be a full modular symbol space given by `msinit` and  $p$  be a prime. To a classical modular symbol  $\phi$  of level  $N$  ( $v_p(N) \leq 1$ ), which is an eigenvector for  $T_p$  with nonzero eigenvalue  $a_p$ , we can attach a  $p$ -adic  $L$ -function  $L_p$ . The function  $L_p$  is defined on continuous characters of  $\text{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})$ ; in GP we allow characters  $\langle \chi \rangle^{s_1} \tau^{s_2}$ , where  $(s_1, s_2)$  are integers,  $\tau$  is the Teichmüller character and  $\chi$  is the cyclotomic character.

The symbol  $\phi$  can be lifted to an *overconvergent* symbol  $\Phi$ , taking values in spaces of  $p$ -adic distributions (represented in GP by a list of moments modulo  $p^n$ ).

`mspadicinit` precomputes data used to lift symbols. If *flag* is given, it speeds up the computation by assuming that  $v_p(a_p) = 0$  if *flag* = 0 (fastest), and that  $v_p(a_p) \geq \textit{flag}$  otherwise (faster as *flag* increases).

`mspadicmoments` computes distributions *mu* attached to  $\Phi$  allowing to compute  $L_p$  to high accuracy.

|   |   |
|---|---|
| initialize $Mp$ to lift symbols               | <code>mspadicinit(M, p, n, {flag})</code>                             |
| lift symbol $\phi$                            | <code>mstooms(Mp, <math>\phi</math>)</code>                           |
| eval overconvergent symbol $\Phi$ on path $p$ | <code>msomseval(Mp, <math>\Phi</math>, <math>p</math>)</code>         |
| $\mu$ for $p$ -adic $L$ -functions            | <code>mspadicmoments(Mp, <math>S</math>, {<math>D = 1</math>})</code> |
| $L_p^{(r)}(\chi^s)$ , $s = [s_1, s_2]$        | <code>mspadicL(mu, {<math>s = 0</math>}, {<math>r = 0</math>})</code> |
| $\hat{L}_p(\tau^i)(x)$                        | <code>mspadicseries(mu, {<math>i = 0</math>})</code>                  |