

# Algebraic Number Theory

(PARI-GP version 2.15.0)

## Binary Quadratic Forms

create  $ax^2 + bxy + cy^2$  **Qfb**( $a, b, c$ ) or **Qfb**( $[a, b, c]$ )  
reduce  $x$  ( $s = \sqrt{D}$ ,  $l = \lfloor s \rfloor$ ) **qfbred**( $x, \{flag\}, \{D\}, \{l\}, \{s\}$ )  
return  $[y, g]$ ,  $g \in \text{SL}_2(\mathbf{Z})$ ,  $y = g \cdot x$  reduced **qfbreds12**( $x$ )  
composition of forms  $x*y$  or **qfbnucomp**( $x, y, l$ )  
 $n$ -th power of form  $x^n$  or **qfbnpow**( $x, n$ )  
composition **qfbcomp**( $x, y$ )  
... without reduction **qfbcomppraw**( $x, y$ )  
 $n$ -th power **qfbpow**( $x, n$ )  
... without reduction **qfbpowraw**( $x, n$ )  
prime form of disc.  $x$  above prime  $p$  **qfbprimeform**( $x, p$ )  
class number of disc.  $x$  **qfbclassno**( $x$ )  
Hurwitz class number of disc.  $x$  **qfbhclassno**( $x$ )  
solve  $Q(x, y) = n$  in integers **qfbsolve**( $Q, n$ )  
solve  $x^2 + Dy^2 = p$ ,  $p$  prime **qfbcornacchia**( $D, p$ )  
...  $x^2 + Dy^2 = 4p$ ,  $p$  prime **qfbcornacchia**( $D, 4 * p$ )

## Quadratic Fields

quadratic number  $\omega = \sqrt{x}$  or  $(1 + \sqrt{x})/2$  **quadgen**( $x$ )  
minimal polynomial of  $\omega$  **quadpoly**( $x$ )  
discriminant of **Q**( $\sqrt{x}$ ) **quaddisc**( $x$ )  
regulator of real quadratic field **quadregulator**( $x$ )  
fundamental unit in  $O_D$ ,  $D > 0$  **quadunit**( $D, \{ 'w \}$ )  
norm of fundamental unit in  $O_D$  **quadunitnorm**( $D$ )  
index of  $O_{Df_2}^\times$  in  $O_D^\times$  **quadunitindex**( $D, f$ )  
class group of **Q**( $\sqrt{D}$ ) **quadclassunit**( $D, \{flag\}, \{t\}$ )  
Hilbert class field of **Q**( $\sqrt{D}$ ) **quadhilbert**( $D, \{flag\}$ )  
... using specific class invariant ( $D < 0$ ) **polclass**( $D, \{inv\}$ )  
ray class field modulo  $f$  of **Q**( $\sqrt{D}$ ) **quadrays**( $D, f, \{flag\}$ )

## General Number Fields: Initializations

The number field  $K = \mathbf{Q}[X]/(f)$  is given by irreducible  $f \in \mathbf{Q}[X]$ . We denote  $\theta = \bar{X}$  the canonical root of  $f$  in  $K$ . A *nf* structure contains a maximal order and allows operations on elements and ideals. A *bnf* adds class group and units. A *bnr* is attached to ray class groups and class field theory. A *rnf* is attached to relative extensions  $L/K$ .

init number field structure *nf* **nfinit**( $f, \{flag\}$ )  
  known integer basis  $B$  **nfinit**( $[f, B]$ )  
  order maximal at  $vp = [p_1, \dots, p_k]$  **nfinit**( $[f, vp]$ )  
  order maximal at all  $p \leq P$  **nfinit**( $[f, P]$ )  
  certify maximal order **nfcertify**(*nf*)

### nf members:

a monic  $F \in \mathbf{Z}[X]$  defining  $K$  **nf.pol**  
number of real/complex places **nf.r1/r2/sign**  
discriminant of *nf* **nf.disc**  
primes ramified in *nf* **nf.p**  
 $T_2$  matrix **nf.t2**  
complex roots of  $F$  **nf.roots**  
integral basis of  $\mathbf{Z}_K$  as powers of  $\theta$  **nf.zk**  
different/codifferent **nf.diff**, **nf.codiff**  
index  $[\mathbf{Z}_K : \mathbf{Z}[X]/(F)]$  **nf.index**  
recompute *nf* using current precision **nfnewprec**(*nf*)  
init relative *rnf*  $L = K[Y]/(g)$  **rnfinit**(*nf*,  $g$ )  
init *bnf* structure **bnfinit**( $f, 1$ )

**bnf members:** same as *nf*, plus  
  underlying *nf* **bnf.nf**  
  class group, regulator **bnf.clgp**, **bnf.reg**  
  fundamental/torsion units **bnf.fu**, **bnf.tu**  
  add  $S$ -class group and units, yield *bnfS* **bnfsunit**(*bnf*,  $S$ )  
  init class field structure *bnr* **bnrinit**(*bnf*,  $m, \{flag\}$ )  
**bnr members:** same as *bnf*, plus  
  underlying *bnf* **bnr.bnf**  
  big ideal structure **bnr.bid**  
  modulus  $m$  **bnr.mod**  
  structure of  $(\mathbf{Z}_K/m)^*$  **bnr.zkst**

## Fields, subfields, embeddings

**Defining polynomials, embeddings**  
(some) number fields with Galois group  $G$  **nflist**( $G$ )  
... and  $|\text{disc}(K)| = N$  and  $s$  complex places **nflist**( $G, N, \{s\}$ )  
... and  $a \leq |\text{disc}(K)| \leq b$  **nflist**( $G, [a, b], \{s\}$ )  
smallest poly defining  $f = 0$  (slow) **polredabs**( $f, \{flag\}$ )  
small poly defining  $f = 0$  (fast) **polredbest**( $f, \{flag\}$ )  
monic integral  $g = Cf(x/L)$  **poltomonic**( $f, \{\&L\}$ )  
random Tschirnhausen transform of  $f$  **poltschirnhaus**( $f$ )  
**Q**[ $t$ ]/( $f$ )  $\subset$  **Q**[ $t$ ]/( $g$ ) ? Isomorphic? **nfisincl**( $f, g$ ), **nfisisom**  
reverse polmod  $a = A(t) \bmod T(t)$  **modreverse**( $a$ )  
compositum of **Q**[ $t$ ]/( $f$ ), **Q**[ $t$ ]/( $g$ ) **polcompositum**( $f, g, \{flag\}$ )  
compositum of  $K[t]/(f)$ ,  $K[t]/(g)$  **nfcompositum**(*nf*,  $f, g, \{flag\}$ )  
splitting field of  $K$  (degree divides  $d$ ) **nfsplitting**(*nf*,  $\{d\}$ )  
signs of real embeddings of  $x$  **nfeltsign**(*nf*,  $x, \{pl\}$ )  
complex embeddings of  $x$  **nfeltembed**(*nf*,  $x, \{pl\}$ )  
 $T \in K[t]$ , # of real roots of  $\sigma(T) \in R[t]$  **nfpolsturm**(*nf*,  $T, \{pl\}$ )

### Subfields, polynomial factorization

subfields (of degree  $d$ ) of *nf* **nfsubfields**(*nf*,  $\{d\}$ )  
maximal subfields of *nf* **nfsubfieldsmax**(*nf*)  
maximal CM subfield of *nf* **nfsubfieldscm**(*nf*)  
 $K_d \subset \mathbf{Q}(\zeta_n)$ , using Gaussian periods **polsubcyclo**( $n, d, \{v\}$ )  
... using class field theory **polsubcyclofast**( $n, d$ )  
roots of unity in *nf* **nfrootsof1**(*nf*)  
roots of  $g$  belonging to *nf* **nfroots**(*nf*,  $g$ )  
factor  $g$  in *nf* **nfactor**(*nf*,  $g$ )

### Linear and algebraic relations

poly of degree  $\leq k$  with root  $x \in \mathbf{C}$  or  $\mathbf{Q}_p$  **algdep**( $x, k$ )  
alg. dep. with pol. coeffs for series  $s$  **seralgdep**( $s, x, y$ )  
diff. dep. with pol. coeffs for series  $s$  **serdiffdep**( $s, x, y$ )  
small linear rel. on coords of vector  $x$  **lindep**( $x$ )

## Basic Number Field Arithmetic (nf)

Number field elements are **t\_INT**, **t\_FRAC**, **t\_POL**, **t\_POLMOD**, or **t\_COL** (on integral basis *nf.zk*).

### Basic operations

$x + y$  **nfeltadd**(*nf*,  $x, y$ )  
 $x \times y$  **nfeltmul**(*nf*,  $x, y$ )  
 $x^n$ ,  $n \in \mathbf{Z}$  **nfeltpow**(*nf*,  $x, n$ )  
 $x/y$  **nfeltdiv**(*nf*,  $x, y$ )  
 $q = x \setminus y := \text{round}(x/y)$  **nfeltdiveuc**(*nf*,  $x, y$ )  
 $r = x \setminus y := x - (x \setminus y)y$  **nfeltmod**(*nf*,  $x, y$ )  
...  $[q, r]$  as above **nfeltdivrem**(*nf*,  $x, y$ )  
reduce  $x$  modulo ideal  $A$  **nfeltreduce**(*nf*,  $x, A$ )  
absolute trace  $\text{Tr}_{K/\mathbf{Q}}(x)$  **nfelttrace**(*nf*,  $x$ )  
absolute norm  $N_{K/\mathbf{Q}}(x)$  **nfeltnorm**(*nf*,  $x$ )

is  $x$  a square? **nfeltissquare**(*nf*,  $x, \{\&y\}$ )  
... an  $n$ -th power? **nfeltispower**(*nf*,  $x, n, \{\&y\}$ )

**Multiplicative structure of  $K^*$ ;  $K^*/(K^*)^n$**   
valuation  $v_{\mathfrak{p}}(x)$  **nfeltval**(*nf*,  $x, \mathfrak{p}$ )  
... write  $x = \pi^{v_{\mathfrak{p}}(x)}y$  **nfeltval**(*nf*,  $x, \mathfrak{p}, \&y$ )  
quadratic Hilbert symbol (at  $\mathfrak{p}$ ) **nfhilbert**(*nf*,  $a, b, \{\mathfrak{p}\}$ )  
 $b$  such that  $xb^n = v$  is small **idealredmodpower**(*nf*,  $x, n$ )

### Maximal order and discriminant

integral basis of field **Q**[ $x$ ]/( $f$ ) **nfbasis**( $f$ )  
field discriminant of **Q**[ $x$ ]/( $f$ ) **nfdisc**( $f$ )  
... and factorization **nfdiscfactors**( $f$ )  
express  $x$  on integer basis **nfalgtobasis**(*nf*,  $x$ )  
express element  $x$  as a polmod **nfbasistoalg**(*nf*,  $x$ )

### Hecke Grossencharacters

Let  $K$  be a number field and  $m$  a modulus. A *gchar* structure describes the group of Hecke Grossencharacters of  $K$  of modulus  $m$  and allows computations with these characters. A character  $\chi$  is described by its components modulo *gc.cyc*.

init *gchar* structure *gc* for modulus  $m$  **gcharinit**(*bnf*,  $m, \{cm\}$ )

### gc members:

  underlying *bnf* **gc.bnf**  
  modulus **gc.mod**  
  elementary divisors (including 0s) **gc.cyc**  
recompute *gc* using current precision **gcharnewprec**(*gc*)  
evaluate Hecke character *chi* at ideal *id* **gchareval**(*gc*, *chi*, *id*)  
exponent column of *id* in  $\mathbf{R}^n$  **gcharideallog**(*gc*, *id*)  
log representation of ideal *id* **gcharlog**(*gc*, *id*)  
... of character  $\chi$  **gcharduallog**(*gc*, *chi*)  
exponent vector of  $\chi$  in  $\mathbf{R}^n$  **gcharparameters**(*gc*, *chi*)  
conductor of  $\chi$  **gcharconductor**(*gc*, *chi*)  
L-function of  $\chi$  **lfuncreate**(*gc*, *chi*)  
local component  $\chi_v$  of  $\chi$  **gcharlocal**(*gc*, *chi*,  $v$ )  
 $\chi$  s.t.  $\chi_v \approx L_{chiv}[i]$  for  $v = Lv[L_{chiv}]$  **gcharidentify**(*gc*,  $Lv, L_{chiv}$ )  
basis of group of algebraic characters **gcharalgebraic**(*gc*)  
is  $\chi$  algebraic? **gcharisalgebraic**(*gc*, *chi*)

### Dedekind Zeta Function $\zeta_K$ , Hecke $L$ series

$R = [c, w, h]$  in initialization means we restrict  $s \in \mathbf{C}$  to domain  $|\Re(s) - c| < w$ ,  $|\Im(s)| < h$ ;  $R = [w, h]$  encodes  $[1/2, w, h]$  and  $[h]$  encodes  $R = [1/2, 0, h]$  (critical line up to height  $h$ ).

$\zeta_K$  as Dirichlet series,  $N(I) \leq b$  **dirzetak**(*nf*,  $b$ )  
init  $\zeta_K^{(k)}(s)$  for  $k \leq n$  **L = lfunitinit**(*bnf*,  $R, \{n = 0\}$ )  
compute  $\zeta_K(s)$  ( $n$ -th derivative) **lfun**( $L, s, \{n = 0\}$ )  
compute  $\Lambda_K(s)$  ( $n$ -th derivative) **lfunlambda**( $L, s, \{n = 0\}$ )

init  $L_K^{(k)}(s, \chi)$  for  $k \leq n$  **L = lfunitinit**( $[bnr, chi], R, \{n = 0\}$ )  
compute  $L_K(s, \chi)$  ( $n$ -th derivative) **lfun**( $L, s, \{n\}$ )  
Artin root number of  $K$  **bnrrootnumber**(*bnr*, *chi*,  $\{flag\}$ )  
 $L(1, \chi)$ , for all  $\chi$  trivial on  $H$  **bnrL1**(*bnr*,  $\{H\}, \{flag\}$ )

## Class Groups & Units (bnf, bnr)

Class field theory data  $a_1, \{a_2\}$  is usually *bnr* (ray class field), *bnr*,  $H$  (congruence subgroup) or *bnr*,  $\chi$  (character on **bnr.clgp**). Any of these define a unique abelian extension of  $K$ .  
units /  $S$ -units **bnfunits**(*bnf*,  $\{S\}$ )  
remove GRH assumption from *bnf* **bnfcertify**(*bnf*)

expo. of ideal $x$ on class gp	<code>bnfisprincipal(bnf, x, {flag})</code>
... on ray class gp	<code>bnrisprincipal(bnr, x, {flag})</code>
expo. of $x$ on fund. units	<code>bnfisunit(bnf, x)</code>
... on $S$ -units, $U$ is <code>bnfunits(bnf, S)</code>	<code>bnfisunit(bnfs, x, U)</code>
signs of real embeddings of <code>bnf.fu</code>	<code>bnfsignunit(bnf)</code>
narrow class group	<code>bnfnarrow(bnf)</code>

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## Primes and Multiplicative Structure

Extension  $L/K$  is defined by  $T \in K[x]$ .

## Lifts and Push-downs

## Norms and Trace

## Factorization

### Maximal order $\mathbf{Z}_L$ as a $\mathbf{Z}_K$ -module

<b>General <math>\mathbf{Z}_K</math>-modules:</b> $M = [\text{matrix, vec. of ideals}] \subset L$	
relative HNF / SNF	<code>nfhnf(nf, M), nfsnf</code>
multiple of det $M$	<code>nfdetint(nf, M)</code>
HNF of $M$ where $d = nf\text{detint}(M)$	<code>nfhnfmod(x, d)</code>
reduced basis for $M$	<code>rnfilllgram(nf, T, M)</code>
determinant of pseudo-matrix $M$	<code>rnfdet(nf, M)</code>
Steinitz class of $M$	<code>rnfsteinitz(nf, M)</code>
$\mathbf{Z}_K$ -basis of $M$ if $\mathbf{Z}_K$ -free, or 0	<code>rnfhnfbasis(bnf, M)</code>
$n$ -basis of $M$ , or $(n+1)$ -generating set	<code>rnfbasis(bnf, M)</code>
is $M$ a free $\mathbf{Z}_K$ -module?	<code>rnfisfree(bnf, M)</code>

Based on an earlier version by Joseph H. Silverman

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Associative Algebras

*A* is a general associative algebra given by a multiplication table *mt* (over **Q** or **F<sub>p</sub>**); represented by *al* from `algtableinit`.  
create *al* from *mt* (over **F<sub>p</sub>**)                    `algtableinit(mt, {p = 0})`  
group algebra **Q**[*G*] (or **F<sub>p</sub>**[*G*])                    `alggroup(G, {p = 0})`  
center of group algebra                    `alggrouppcenter(G, {p = 0})`  
**Properties**  
is (*mt*, *p*) OK for `algtableinit`?                    `algisassociative(mt, {p = 0})`  
multiplication table *mt*                    `algmultable(al)`  
dimension of *A* over prime subfield                    `algdim(al)`  
characteristic of *A*                    `algchar(al)`  
is *A* commutative?                    `algiscommutative(al)`  
is *A* simple?                    `algissimple(al)`  
is *A* semi-simple?                    `algissemisimple(al)`  
center of *A*                    `algcenter(al)`  
Jacobson radical of *A*                    `algradical(al)`  
radical *J* and simple factors of *A*/*J*                    `algsimpledec(al)`  
**Operations on algebras**  
create *A*/*I*, *I* two-sided ideal                    `algquotient(al, I)`  
create *A*<sub>1</sub> ⊗ *A*<sub>2</sub>                    `algtensor(al1, al2)`  
create subalgebra from basis *B*                    `algsubalg(al, B)`  
quotients by ortho. central idempotents *e*                    `algcentralproj(al, e)`  
isomorphic alg. with integral mult. table                    `algmakeintegral(mt)`  
prime subalgebra of semi-simple *A* over **F<sub>p</sub>**                    `algprimesubalg(al)`  
find isomorphism *A* ≅ *M<sub>d</sub>*(**F<sub>q</sub>**)                    `algsplit(al)`  
**Operations on lattices in algebras**  
lattice generated by cols. of *M*                    `alglathnf(al, M)`  
... by the products *xy*, *x* ∈ *lat1*, *y* ∈ *lat2*                    `alglatmul(al, lat1, lat2)`  
sum *lat1* + *lat2* of the lattices                    `alglatadd(al, lat1, lat2)`  
intersection *lat1* ∩ *lat2*                    `alglatinter(al, lat1, lat2)`  
test *lat1* ⊂ *lat2*                    `alglatsubset(al, lat1, lat2)`  
generalized index (*lat2* : *lat1*)                    `alglatindex(al, lat1, lat2)`  
{*x* ∈ *al* | *x* · *lat1* ⊂ *lat2*}                    `alglatlefttransporter(al, lat1, lat2)`  
{*x* ∈ *al* | *lat1* · *x* ⊂ *lat2*}                    `alglatrighttransporter(al, lat1, lat2)`  
test *x* ∈ *lat* (set *c* = coord. of *x*)                    `alglatcontains(al, lat, x, {&c})`  
element of *lat* with coordinates *c*                    `alglatelement(al, lat, c)`  
**Operations on elements**  
*a* + *b*, *a* − *b*, −*a*                    `algadd(al, a, b), algsub, algneg`  
*a* × *b*, *a*<sup>2</sup>                    `algmul(al, a, b), algsqr`  
*a<sup>n</sup>*, *a*<sup>−1</sup>                    `algpow(al, a, n), alginv`  
is *x* invertible ? (then set *z* = *x*<sup>−1</sup>)                    `alginv(al, x, {&z})`  
find *z* such that *x* × *z* = *y*                    `algdivl(al, x, y)`  
find *z* such that *z* × *x* = *y*                    `algdivr(al, x, y)`  
does *z* s.t. *x* × *z* = *y* exist? (set it)                    `algsdivl(al, x, y, {&z})`  
matrix of *v* ↦ *x* · *v*                    `algtomatrix(al, x)`  
absolute norm                    `algnorm(al, x)`  
absolute trace                    `algtrace(al, x)`  
absolute char. polynomial                    `algcharpoly(al, x)`  
given *a* ∈ *A* and polynomial *T*, return *T*(*a*)                    `algpoleval(al, T, a)`  
random element in a box                    `algrandom(al, b)`

Central Simple Algebras

*A* is a central simple algebra over a number field *K*; represented by *al* from `algininit`; *K* is given by a *nf* structure.  
create CSA from data                    `algininit(B, C, {v}, {maxord = 1})`  
multiplication table over *K*                    *B* = *K*, *C* = *mt*  
cyclic algebra (*L*/*K*, *σ*, *b*)                    *B* = *rnf*, *C* = [*sigma*, *b*]  
quaternion algebra (*a*, *b*)<sub>*K*</sub>                    *B* = *K*, *C* = [*a*, *b*]  
matrix algebra *M<sub>d</sub>*(*K*)                    *B* = *K*, *C* = *d*  
local Hasse invariants over *K*                    *B* = *K*, *C* = [*d*, [*PR*, *HF*], *HI*]

Properties

type of *al* (*mt*, CSA)                    `algtype(al)`  
dimension of *A* over **Q**                    `algdim(al, 1)`  
dimension of *al* over its center *K*                    `algdim(al)`  
degree of *A* (= √dim<sub>*K*</sub> *A*)                    `algdegree(al)`  
*al* a cyclic algebra (*L*/*K*, *σ*, *b*); return *σ*                    `algaut(al)`  
...return *b*                    `algb(al)`  
...return *L*/*K*, as an *rnf*                    `algsplittingfield(al)`  
split *A* over an extension of *K*                    `algsplittingdata(al)`  
splitting field of *A* as an *rnf* over center                    `algsplittingfield(al)`  
multiplication table over center                    `algrelmultable(al)`  
places of *K* at which *A* ramifies                    `algramifiedplaces(al)`  
Hasse invariants at finite places of *K*                    `alghassef(al)`  
Hasse invariants at infinite places of *K*                    `alghassei(al)`  
Hasse invariant at place *v*                    `alghasse(al, v)`  
index of *A* over *K* (at place *v*)                    `algindex(al, {v})`  
is *al* a division algebra? (at place *v*)                    `algsdivision(al, {v})`  
is *A* ramified? (at place *v*)                    `algsiramified(al, {v})`  
is *A* split? (at place *v*)                    `algsisplit(al, {v})`

Operations on elements

reduced norm                    `algnorm(al, x)`  
reduced trace                    `algtrace(al, x)`  
reduced char. polynomial                    `algcharpoly(al, x)`  
express *x* on integral basis                    `algalgtobasis(al, x)`  
convert *x* to algebraic form                    `algbasistoalg(al, x)`  
map *x* ∈ *A* to *M<sub>d</sub>*(*L*), *L* split. field                    `algtomatrix(al, x)`

Orders

**Z**-basis of order *O*<sub>0</sub>                    `algbasis(al)`  
discriminant of order *O*<sub>0</sub>                    `algdisc(al)`  
**Z**-basis of natural order in terms *O*<sub>0</sub>'s basis                    `alginvbasis(al)`